IMO Shortlist 1999

Algebra

1 Let $n \geq 2$ be a fixed integer. Find the least constant C such the inequality

$$\sum_{i<j} x_i x_j \left(x_i^2 + x_j^2\right) \leq C \left(\sum_i x_i\right)^4$$

holds for any $x_1, \ldots, x_n \geq 0$ (the sum on the left consists of $\binom{n}{2}$ summands). For this constant C, characterize the instances of equality.

2 The numbers from 1 to n^2 are randomly arranged in the cells of a $n \times n$ square ($n \geq 2$). For any pair of numbers situated on the same row or on the same column the ration of the greater number to the smaller number is calculated. Let us call the **characteristic** of the arrangement the smallest of these $n^2(n-1)$ fractions. What is the highest possible value of the characteristic ?

3 A game is played by n girls ($n \geq 2$), everybody having a ball. Each of the $\binom{n}{2}$ pairs of players, is an arbitrary order, exchange the balls they have at the moment. The game is called nice **nice** if at the end nobody has her own ball and it is called **tiresome** if at the end everybody has her initial ball. Determine the values of n for which there exists a nice game and those for which there exists a tiresome game.

4 Prove that the set of positive integers cannot be partitioned into three nonempty subsets such that, for any two integers x, y taken from two different subsets, the number $x^2 - xy + y^2$ belongs to the third subset.

5 Find all the functions $f : \mathbb{R} \mapsto \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

6 For $n \geq 3$ and $a_1 \leq a_2 \leq \ldots \leq a_n$ given real numbers we have the following instructions:

- place out the numbers in some order in a ring; - delete one of the numbers from the ring; - if just two numbers are remaining in the ring: let S be the sum of these two numbers. Otherwise, if there are more the two numbers in the ring, replace

Afterwards start again with the step (2). Show that the largest sum S which can result in this way is given by the formula

$$S_{max} = \sum_{k=2}^{n} \binom{n-2}{[\frac{k}{2}]-1} a_k.$$

Combinatorics

1 Let $n \geq 1$ be an integer. A **path** from $(0,0)$ to (n,n) in the xy plane is a chain of consecutive unit moves either to the right (move denoted by E) or upwards (move denoted by N), all the moves being made inside the half-plane $x \geq y$. A **step** in a path is the occurence of two consecutive moves of the form EN. Show that the number of paths from $(0,0)$ to (n,n) that contain exactly s steps $(n \geq s \geq 1)$ is

$$\frac{1}{s}\binom{n-1}{s-1}\binom{n}{s-1}.$$

2 If a $5 \times n$ rectangle can be tiled using n pieces like those shown in the diagram, prove that n is even. Show that there are more than $2 \cdot 3^{k-1}$ ways to file a fixed $5 \times 2k$ rectangle $(k \geq 3)$ with $2k$ pieces. (symmetric constructions are supposed to be different.)

3 A biologist watches a chameleon. The chameleon catches flies and rests after each catch. The biologist notices that:

- the first fly is caught after a resting period of one minute; - the resting period before catching the $2m^{th}$ fly is the same as the resting period before catching the m^{th} fly and one minute shorter than the resting period before catching the $(2m+1)^{th}$ fly; - when the chameleon stops resting, he catches a fly instantly.

- How many flies were caught by the chameleon before his first resting period of 9 minutes in a row ? - After how many minutes will the chameleon catch his 98^{th} fly ? - How many flies were caught by the chameleon after 1999 minutes have passed ?

4 Let A be a set of N residues $\pmod{N^2}$. Prove that there exists a set B of of N residues $\pmod{N^2}$ such that $A + B = \{a + b | a \in A, b \in B\}$ contains at least half of all the residues $\pmod{N^2}$.

5 Let b be an even positive integer. We say that two different cells of a $n \times n$ board are **neighboring** if they have a common side. Find the minimal number of cells on he $n \times n$ board that must be marked so that any cell marked or not marked) has a marked neighboring cell.

6 Suppose that every integer has been given one of the colours red, blue, green or yellow. Let x and y be odd integers so that $|x| \neq |y|$. Show that there are two integers of the same colour whose difference has one of the following values: $x, y, x+y$ or $x-y$.

7 Let $p > 3$ be a prime number. For each nonempty subset T of $\{0, 1, 2, 3, \ldots, p-1\}$, let $E(T)$ be the set of all $(p-1)$-tuples $(x_1, \ldots, x_{p-1})$, where each $x_i \in T$ and $x_1 + 2x_2 + \ldots + (p-1)x_{p-1}$ is divisible by p and let $|E(T)|$ denote the number of elements in $E(T)$. Prove that

$$|E(\{0, 1, 3\})| \geq |E(\{0, 1, 2\})|$$

with equality if and only if $p = 5$.

Geometry

1 Let ABC be a triangle and M be an interior point. Prove that

$$\min\{MA, MB, MC\} + MA + MB + MC < AB + AC + BC.$$

2 A circle is called a **separator** for a set of five points in a plane if it passes through three of these points, it contains a fourth point inside and the fifth point is outside the circle. Prove that every set of five points such that no three are collinear and no four are concyclic has exactly four separators.

3 A set S of points from the space will be called **completely symmetric** if it has at least three elements and fulfills the condition that for every two distinct points A and B from S, the perpendicular bisector plane of the segment AB is a plane of symmetry for S. Prove that if a completely symmetric set is finite, then it consists of the vertices of either a regular polygon, or a regular tetrahedron or a regular octahedron.

4 For a triangle $T = ABC$ we take the point X on the side (AB) such that $AX/AB = 4/5$, the point Y on the segment (CX) such that $CY = 2YX$ and, if possible, the point Z on the ray $(CA$ such that $\widehat{CXZ} = 180 - \widehat{ABC}$. We denote by Σ the set of all triangles T for which $\widehat{XYZ} = 45$. Prove that all triangles from Σ are similar and find the measure of their smallest angle.

5 Let ABC be a triangle, Ω its incircle and $\Omega_a, \Omega_b, \Omega_c$ three circles orthogonal to Ω passing through $(B, C), (A, C)$ and (A, B) respectively. The circles Ω_a and Ω_b meet again in C'; in the same way we obtain the points B' and A'. Prove that the radius of the circumcircle of $A'B'C'$ is half the radius of Ω.

6 Two circles Ω_1 and Ω_2 touch internally the circle Ω in M and N and the center of Ω_2 is on Ω_1. The common chord of the circles Ω_1 and Ω_2 intersects Ω in A and B. MA and MB intersects Ω_1 in C and D. Prove that Ω_2 is tangent to CD.

7 The point M is inside the convex quadrilateral $ABCD$, such that

$$MA = MC, \quad \widehat{AMB} = \widehat{MAD} + \widehat{MCD} \quad \text{and} \quad \widehat{CMD} = \widehat{MCB} + \widehat{MAB}.$$

Prove that $AB \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

8 Given a triangle ABC. The points A, B, C divide the circumcircle Ω of the triangle ABC into three arcs BC, CA, AB. Let X be a variable point on the arc AB, and let O_1 and O_2 be the incenters of the triangles CAX and CBX. Prove that the circumcircle of the triangle XO_1O_2 intersects the circle Ω in a fixed point.

IMO Shortlist 1999

Number Theory

1 Find all the pairs of positive integers (x, p) such that p is a prime, $x \leq 2p$ and x^{p-1} is a divisor of $(p-1)^x + 1$.

2 Prove that every positive rational number can be represented in the form $\dfrac{a^3 + b^3}{c^3 + d^3}$ where a,b,c,d are positive integers.

3 Prove that there exists two strictly increasing sequences (a_n) and (b_n) such that $a_n(a_n + 1)$ divides $b_n^2 + 1$ for every natural n.

4 Denote by S the set of all primes such the decimal representation of $\frac{1}{p}$ has the fundamental period divisible by 3. For every $p \in S$ such that $\frac{1}{p}$ has the fundamental period $3r$ one may write

$$\frac{1}{p} = 0, a_1 a_2 \dots a_{3r} a_1 a_2 \dots a_{3r} \dots,$$

where $r = r(p)$; for every $p \in S$ and every integer $k \geq 1$ define $f(k, p)$ by

$$f(k, p) = a_k + a_{k+r(p)} + a_{k+2.r(p)}$$

a) Prove that S is infinite. b) Find the highest value of $f(k, p)$ for $k \geq 1$ and $p \in S$

5 Let n, k be positive integers such that n is not divisible by 3 and $k \geq n$. Prove that there exists a positive integer m which is divisible by n and the sum of its digits in decimal representation is k.

6 Prove that for every real number M there exists an infinite arithmetic progression such that:

- each term is a positive integer and the common difference is not divisible by 10 - the sum of the digits of each term (in decimal representation) exceeds M.